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Duality in 2D Spin Models on Torus

Anatolij I. Bugrij [1]

Bogolyubov Institute for Theoretical Physics

252143, Kiev, Ukraine

Vitalij N. Shadura

Institute of Theoretical and Experimental Physics

117259, Moscow, Russia

and

Bogolyubov Institute for Theoretical Physics

252143, Kiev, Ukraine

Abstract

Method of derivation of the duality relations for two-dimensional $Z(N)$-symmetric spin models on finite square lattice wrapped on the torus is proposed. As example, exact duality relations for the nonhomogeneous Ising model ($N = 2$) and the $Z(N)$-Berezinsky-Villain model are obtained.

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[1] e-mail: abugrij@gluk.apc.org

1 Introduction

Study of duality properties in statistical mechanics and quantum field theory models is important method for non-perturbative investigation of their phase diagram and field content. Duality transformation was discovered by Kramers and Wannier [1] in the two-dimensional Ising model. Kadanoff and Ceva [2] generalized the Kramers-Wannier duality relation to the nonhomogeneous case (the coupling constants are arbitrary functions of lattice site coordinates) with spherical boundary conditions:

$$(\prod_{\widetilde{r},\mu} \sinh 2\widetilde{K}_\mu(\widetilde{r}))^{-1/4} \widetilde{Z}[\widetilde{K}] = (\prod_{r,\mu} \sinh 2K_\mu(r))^{-1/4} Z[K], \quad \mu = x, y, \tag{1}$$

$$\sinh 2K_x(r) \cdot \sinh 2\widetilde{K}_{-y}(\widetilde{r}) = 1, \quad \sinh 2K_y(r) \cdot \sinh 2\widetilde{K}_{-x}(\widetilde{r}) = 1. \tag{2}$$

We denote site coordinates, functions and functionals on the dual lattice by "tilda" : $\widetilde{r}$, $\widetilde{\sigma}(\widetilde{r})$, $\widetilde{K}_\mu(\widetilde{r})$, $\widetilde{H}[\widetilde{K}, \widetilde{\sigma}]$, $\widetilde{Z}[\widetilde{K}]$, ... A site coordinate on the dual lattice coincides with a coordinate of the plaquet center on the original lattice: $\widetilde{r} = r + (\widehat{x} + \widehat{y})/2$ and coupling constants $\widetilde{K}_{-\nu}(\widetilde{r}) = \widetilde{K}_\nu(\widetilde{r} - \widehat{\nu})$ ($\widehat{\nu} = \widehat{x}$, $\widehat{y}$ are the unit vectors along the horisontal X and vertical Y axes).

As was already mentioned in [1,2], relation (1) can not be understood literally. So, for example, using the method of comparing high- and low-temperature expansions for deriving duality relation (1) in the case of the periodical boundary conditions, it is hard to take into account and to compare the graphs wrapping up the torus. In fact (1) is correct in the thermodynamic limit (for the specific free energy). However for the nonhomogeneous case the procedure of thermodynamic limit is rather ambiguous. In [2] this duality relation was obtained for spherical (nonphysical for the lattice) boundary conditions.

In [3], using global Bianchi identities for link formulation of the lattice spin systems on the hypertorus, the contributions of the link variables on the topological nontrivial loops on the hypertorus was selected in the partition function, but the duality relations was not formulated in obvious form in this case.

Since duality is a popular method of non-perturbative investigation in quantum field theory and statistical mechanics (for review see [10]), it is important to formulate a duality transformation for finite systems. Recently, we have suggested [4,5] exact duality relations for the nonhomogeneous Ising model on a finite square lattice of size $n \times m$ wrapped on the torus:

$$\prod_{\widetilde{r},\mu} (\sinh 2\widetilde{K}_\mu(\widetilde{r}))^{-1/4} \widetilde{Z}^{(\widetilde{p}_x,\widetilde{p}_y)}[\widetilde{K}] = \frac{1}{2} \prod_{r,\mu} (\sinh 2K_\mu(r))^{-1/4} \sum_{p_x,p_y=0}^{1} T^{\widetilde{p}_x,\widetilde{p}_y}_{p_x,p_y} Z^{(p_x,p_y)}[K], \tag{3}$$

Here $Z^{(p_x,p_y)}[K]$ are partition functions of the Ising model with corresponding combinations of the periodical ($p_x, p_y = 0$) and antiperiodical ($p_x, p_y = 1$) boundary conditions along the horizontal X and vertical Y axes:

$$Z^{(p_x,p_y)}[K] = \sum_{[\sigma]} \exp(\sum_{r,\nu} K_\nu(r)\sigma(r)\nabla_\nu^{(p_\nu)}\sigma(r)), \tag{4}$$

and

$$\widehat{T} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{5}$$

where $r = (x, y)$ denotes the site coordinates on the square lattice of size $n \times m$, $x = 1, \ldots, n$ $y = 1, \ldots, m$; $\sigma(r) = \pm 1$; $K_x(r)$ and $K_y(r)$ are the coupling constants along corresponding axes. The one-step shift operators ∇_x, ∇_y act on $\sigma(r)$ in the following way

$$\nabla_x \sigma(r) = \sigma(r + \widehat{x}), \quad \nabla_y \sigma(r) = \sigma(r + \widehat{y}). \tag{6}$$

For the periodical (antiperiodical) boundary conditions along X and Y axes we have

$$\nabla_x^{(p_x)}\sigma(n, y) = (-)^{p_x}\sigma(1, y), \quad \nabla_y^{(p_y)}\sigma(x, m) = (-)^{p_y}\sigma(x, 1). \tag{7}$$

In Ref. [4] the duality relation (4) was proved for homogeneous and weakly nonhomogeneous distributions of the coupling constants. We also have checked the duality relation (4) for lattices of small sizes by direct calculation on the computer. As a corollary of (4), we obtained [4,5] the duality relations for the two-point correlation function on the torus, for the partition functions of the 2D Ising model with magnetic fields applied to the boundaries and the 2D Ising model with free, fixed and mixed boundary conditions.

In this paper we formulate method of derivation of the duality relations for two-dimensional $Z(N)$-symmetric spin models on finite square lattice wrapped on the torus. As example, the duality relations for the vector Potts model (the $N = 2$ case is considered in detail) and the $Z(N)$-Berezinsky-Villain model [6,7] are obtained. Without taking account of boundary conditions duality relations for these models was obtained in [8,9] (for review see [10]). In principle suggested method it is not hard to generalize for lattices with larger dimensions compactified on the hypertorus and the lattice models with continious global or gauge symmetries.

To formulate the method let us introduce definition of magnetic dislocations connected with boundary conditions, "topological" charge of dislocation and gauge transformations of

coupling constant configurations for the vector Potts model. The hamiltonian of this model one can write in the following form:

$$-\beta H_V^{(p,q)}[K,\sigma] = \frac{1}{2}\sum_{r,\nu}(K_\nu(r)\sigma^*(r)\nabla_\nu^{(p_\nu)}\sigma(r) + \text{c.c.}) \tag{8}$$

where a spin variable takes N values: $\sigma(r) = \exp(i\frac{2\pi}{N}l(r))$, $\quad l(r) = 0, ..., N-1$, $\nu = x, y$ and $p_x = p$, $p_y = q$ ($p, q = 0, ..., N-1$) designate the cyclic boundary conditions for one-step shift operators (6):

$$\nabla_x^{(p)}\sigma(n,y) = e^{i\frac{2\pi}{N}p}\sigma(1,y), \quad \nabla_y^{(q)}\sigma(x,m) = e^{i\frac{2\pi}{N}q}\sigma(x,1). \tag{9}$$

These conditions have the following form for variable $l(r)$:

$$l(n+1,y) = l(1,y) + p, \quad l(x,m+1) = l(x,1) + q. \tag{10}$$

For the periodical boundary conditions we have $p = 0$ and $q = 0$.

Using (9), Hamiltonian $H^{(p,q)}[K,\sigma]$ one can write as Hamiltonian $H_D^{(0,0)}[K,d,\sigma]$ with the magnetic dislocation $D^{(p,q)}$ corresponding boundary conditions (p,q) and with periodical boundary conditions for spin variable $\sigma(r)$:

$$-\beta H^{(p,q)}[K,\sigma] = -\beta H_D^{(0,0)}[K,d,\sigma] = \frac{1}{2}\sum_{r,\nu}[K_\nu(r)\exp(i\frac{2\pi}{N}d_\nu^{(p,q)}(r))\,\sigma^*(r)\nabla_\nu^{(0)}\sigma(r) + \text{c.c.}]$$

$$= \sum_{r,\nu} K_\nu(r)\cos\frac{2\pi}{N}(\Delta_\nu l(r) + d_\nu^{(p,q)}(r)), \tag{11}$$

Here $\Delta_\nu = \nabla_\nu^{(0)} - 1$ is difference derivative with the periodical boundary conditions, vector fields $K_\nu(r)$ and $d_\nu^{(p,q)}(r)$, determined on the lattice bonds, it is convenient to consider as the module and the phase of the complex coupling constant. The magnetic dislocation $D^{(p,q)}$ is determined by the phase

$$d_\nu^{(p,q)}(r) = (d_x(r), d_y(r)) = (p\delta_{B_X}(r), q\delta_{B_Y}(r)\,), \tag{12}$$

which is unequal zero along the boundary cycle B_X and B_Y, setting the space configuration of the dislocation on the torus:

$$\delta_{B_X}(r) = \sum_{r' \in B_X}\delta^2(r-r'), \quad \delta_{B_Y}(r) = \sum_{r' \in B_Y}\delta^2(r-r'), \tag{13}$$

where $\delta^2(r-r')$ is Kronecker δ-function and

$$B_X = \{(x,m), x = 1, ..., n\}\,, \quad B_Y = \{(n,y), y = 1, ..., m\}\,.$$

The phase $d_\nu^{(p,q)}(r)$ one can consider as density of a "topological" charge Q_ν of the magnetic dislocation. This charge, for example, for dislocation $D^{(p,q)}$ is equal

$$Q_\nu = \sum_r d_\nu^{(p,q)}(r) = (pn, qm). \tag{14}$$

We will call magnetic dislocations $D^{(p,q)}$ $(p, q = 0, ..., N-1)$ as basic magnetic dislocations. Note that periodical boundary conditions ($p = q = 0$) along all cycles of the torus correspond the absence of the magnetic dislocations. Nevertheless, for convenience we have introduce denotion $D^{(0,0)}$ for this case.

Hamiltonian (8) has invariance relative to Z_N-gauge transformations [11]

$$\sigma'(r) = e^{i\frac{2\pi}{N}\phi(r)}\sigma(r), \quad K'_\mu(r) = e^{i\frac{2\pi}{N}\phi(r)}K_\mu(r)e^{i\frac{2\pi}{N}\phi(r+\hat{\mu})}, \tag{15}$$

where $\phi(r)$ has the periodical boundary conditions. This invariance gives the following relation for partition function:

$$Z_V^{(p,q)}[K] = \sum_{[\sigma]} e^{-\beta H^{(p,q)}[K,\sigma]} = \sum_{[\sigma']} e^{-\beta H^{(p,q)}[K',\sigma']} = Z_V^{(p,q)}[K'].$$

Note that the gauge transformation of $l(r)$ and $d_\mu(r)$ in Hamiltonian (11) has form:

$$l'(r) = l(r) + \phi(r), \quad d'_\mu(r) = d_\mu^{(p,q)}(r) + \Delta_\mu\phi(r). \tag{16}$$

These transformations lead to both the deformation of the basic magnetic dislocations and the appearance of new closed dislocations. Then $d'_\mu(r)$ describes the field of closed magnetic dislocations on the torus. It is obvious that the topological charge does not change at the gauge transformation. For example, for Hamiltonian $H_D^{(0,0)}[K, d, \sigma]$ with dislocation $D^{(p,q)}$ we have

$$Q'_\mu = \sum_r d_\mu^{(g)}(r) = \sum_r d_\mu^{(p,q)}(r) + \sum_r \Delta_\mu\phi(r).$$

Here the periodical boundary conditions for $\phi(r)$ lead to vanishing of the second term and $Q'_\mu = Q_\mu$. From here it follows that the set of coupling constant configurations $\{[K, d^{(g)}]\}$ (contained closed dislocations) one can divide on the gauge-nonequivalent classes $\Omega^{(p,q)}$ with corresponding value of topological charge $Q_\mu = (pn, qm)$. Elements of class $\Omega^{(p,q)}$ one can generate with help of gauge transformations (15) from the basic magnetic dislocation $D^{(p,q)}$.

Let us briefly formulate the idea of suggested method. Using the Fourier transformation method for derivation of the duality relations, we obtain the expression with δ-functions. The solution of the corresponding system of equations defines the relation between the initial and dual spin variable. Usually, for example, see [10], omitting the problem of taking account of boundary conditions, the only one solution of this system of equations is written. However for the lattice model on the torus we can find many solutions of this system. These solutions one can classify over the gauge-nonequivalent classes $\tilde{\Omega}^{(\tilde{p},\tilde{q})}$ of coupling constant configurations for the dual model and also each class has definite value of topological charge $\tilde{Q}_\mu = (\tilde{p}n, \tilde{q}m)$. Therefore at dual transformation of the partition function it is necessary to sum over all the gauge-nonequivalent classes on the dual lattice with coefficients, depending from boundary conditions on the initial lattice.

2 Vector Potts model

Now, using the method discussed in previous section, we derive the duality relation for the vector Potts model. Partition function (11) of this model one can represented in the following form [9,10]:

$$Z_V^{(p,q)}[K,d] = \sum_{[l]} \exp\left\{-\beta H^{(p,q)}[K,l]\right\} = \sum_{[l]} \exp\left\{\sum_{r,\mu} K_\mu(r) \cos\frac{2\pi}{N}(\Delta_\mu l(r) + d_\mu^{(p,q)}(r))\right\} = \tag{17}$$

$$\sum_{[l]}\sum_{[t]} \exp(-\beta\tilde{H}[t]) \exp\left\{i\frac{2\pi}{N}\sum_{r,\mu} t_\mu(r)(\Delta_\mu l(r) + d_\mu^{(p,q)}(r))\right\} = \tag{18}$$

$$\sum_{[t]} \exp(-\beta\tilde{H}[t] + i\frac{2\pi}{N}\sum_{r,\mu} t_\mu(r) d_\mu^{(p,q)}(r)) \prod_r N\delta_N(\Delta_\mu t_\mu(r-\widehat{\mu})), \tag{19}$$

where

$$\sum_{[l]} = \prod_r (\sum_{l(r)=0}^{N-1}), \quad \sum_{[t]} = \prod_r (\sum_{t_\mu(r)=0}^{N-1}).$$

In (18) we have made the Fourier transformation to vector field $t_\mu(r)$ ($t_\mu(r) = 0, 1, ..., N-1$). $-\beta\tilde{H}[t]$ is Fourier-transform of Hamiltonian (11):

$$-\beta\tilde{H}[t] = \sum_{k=0}^{M}\sum_{r,\mu} g_\mu^{(k)}(K) \cos^k \frac{2\pi}{N} t_\mu(r). \tag{20}$$

Here $M = N/2$, if N is even and $M = (N-1)/2$, if N is odd. In (19) $\delta_N(s)$ is Kronecker δ_N-function mod N: it is one if $s = NL$, where L is integer and zero in another case.

In order to get rid of δ_N-function in (19) it is necessary to solve equation

$$\Delta_\mu t_\mu(r-\widehat{\mu}) = 0 \quad /. \text{ mod N}. \tag{21}$$

Nontrivial solutions of this equation on the torus one can write in the form:

$$t_\mu^{(\alpha)}(r) = \epsilon_{\mu\nu}\Delta_\nu \tilde{l}(\tilde{r}-\widehat{\nu}) + \epsilon_{\mu\nu}\tilde{d}_\nu^{(\alpha)}(\tilde{r}-\widehat{\nu}), \tag{22}$$

where index α numerates the solutions, $\tilde{l}(\tilde{r}) = 0, 1, ..., N-1$ is defined on a site of the dual lattice and $\tilde{d}_\nu^{(\alpha)}(\tilde{r})$ is density of the topological charge (corresponding given solution α) of the field of the closed magnetic dislocations on the dual lattice

$$\tilde{d}_\mu^{(\alpha)}(\tilde{r}) = \sum_{i\in Z_\alpha} s_i^{(\alpha)} \sum_{r'\in\Gamma_i} \epsilon_{\mu\nu} a_\nu(r')\delta^2(r-r'), \quad s_i^{(\alpha)} = 0, 1, ..., N-1. \tag{23}$$

Here by analogy with (12), (13) $\tilde{d}_\mu^{(\alpha)}(\tilde{r})$ is defined on bonds of the dual lattice. For convenience we have written the dislocations on the dual lattice by means of closed paths Γ_i on the original

lattice. Z_α in (23) denotes subset of the paths (corresponding to solution α) from set Γ of all closed paths on original lattice ($\Gamma_i \in \Gamma$). Vector $a_\mu(r) = e_\mu(r)$ if the direction of circuit over path Γ_i in site r (the direction of circuit is counterclockwise) coincides with direction of the unit vector $e_\mu(r) = \widehat{\mu}$ in this site, otherwise $a_\mu(r) = -e_\mu(r)$.

Expression (23) it is not hard to obtain, observing, that solution (22) satisfies by equation (21) on site $\tilde{r}$ when

$$\epsilon_{\mu\nu}\Delta_\mu \tilde{d}_\nu^{(\alpha)}(\tilde{r} - \widehat{\mu} - \widehat{\nu}) = 0.$$

This equation becomes the identity if m the following conditions are fulfilled:

$$\tilde{d}_y^{(\alpha)}(\tilde{r} - \widehat{y}) = \tilde{d}_y^{(\alpha)}(\tilde{r} - \widehat{x} - \widehat{y}), \quad \tilde{d}_x^{(\alpha)}(\tilde{r} - \widehat{x}) = \tilde{d}_x^{(\alpha)}(\tilde{r} - \widehat{x} - \widehat{y});$$

$$\tilde{d}_y^{(\alpha)}(\tilde{r} - \widehat{y}) = \tilde{d}_x^{(\alpha)}(\tilde{r} - \widehat{x}), \quad \tilde{d}_y^{(\alpha)}(\tilde{r} - \widehat{x} - \widehat{y}) = \tilde{d}_x^{(\alpha)}(\tilde{r} - \widehat{x} - \widehat{y});$$

$$\tilde{d}_y^{(\alpha)}(\tilde{r} - \widehat{y}) = -\tilde{d}_x^{(\alpha)}(\tilde{r} - \widehat{x} - \widehat{y}), \quad \tilde{d}_x^{(\alpha)}(\tilde{r} - \widehat{x}) = \tilde{d}_y^{(\alpha)}(\tilde{r} - \widehat{x} - \widehat{y}).$$

Consistency of these solutions on some set of sites requires that these sites belong to closed paths Γ_i on the torus, that is these solutions must be "glued" in order to form the closed magnetic dislocatuons.

Let us denote by $[\tilde{d}^{(\alpha)}]$ coupling constant configurations on the dual lattice corresponding to the solution (23). Depending on the number of the solution these configurations cointain both the closed dislocations non-enveloping of the cycles of the torus and the dislocations enveloping of ones. The dislocations of the first type remove by means of gauge transformations (16) on the dual lattice and the dislocations of the second type can be trasformed to the basic magnetic dislocations $\tilde{D}^{(\widetilde{p},\widetilde{q})}$. This means that all configurations $[\tilde{d}^{(\alpha)}]$ can be classified in the gauge-nonequivalent classes $\tilde{\Omega}^{(\widetilde{p},\widetilde{q})}$ with topological charge

$$\tilde{Q}_\nu = \sum_{\tilde{r}} \tilde{d}_\nu^{(\widetilde{p},\widetilde{q})}(\tilde{r}) = (\tilde{p}n, \tilde{q}m),$$

where $\widetilde{p}, \widetilde{q} = 0, 1, \ ..., N-1$.

Since the duality relation connects the partition functions, which are the gauge-invariant quantities, at removal δ_N-functions in (19) we must sum over the gauge-nonequivalent solutions of equation (21):

$$t_\mu^{(\widetilde{p},\widetilde{q})}(r) = \epsilon_{\mu\nu}\Delta_\nu \widetilde{l}(\tilde{r} - \widehat{\nu}) + \epsilon_{\mu\nu}\tilde{d}_\nu^{(\widetilde{p},\widetilde{q})}(\tilde{r} - \widehat{\nu}), \tag{24}$$

where $\tilde{d}_\mu^{(\widetilde{p},\widetilde{q})}$ is defined on the dual lattice by relations similar to (12)-(14). Substituting these solutions in (19), we obtain

$$Z_V^{(p,q)}[K, d] = \frac{1}{N} \sum_{\widetilde{p},\widetilde{q}} \sum_{[\tilde{l}]} \exp(-\beta \tilde{H}[\Delta_\mu \tilde{l} + \tilde{d}_\mu^{(\tilde{p},\tilde{q})}])$$

$$\exp\{i\frac{2\pi}{N}\sum_{r,\mu}\epsilon_{\mu\nu}d_\mu^{(p,q)}(r)[\Delta_\nu\tilde{l}(\tilde{r}-\hat{\nu})+\tilde{d}_\nu^{(\widetilde{p},\widetilde{q})}(\tilde{r}-\hat{\nu})]\}.$$

Here we have introduced factor $1/N$ as taking into account relation (24), it is not hard to note that the sum over configurations $[l]$ in N times more than the sum over $[t]$. Remarking, that

$$\sum_{r,\mu}\epsilon_{\mu\nu}d_\mu^{(p,q)}(r)\Delta_\nu\tilde{l}(\tilde{r}-\hat{\nu})=0,$$

relation (24) one can write in compact form

$$Z_V^{(p,q)}[K,d]=\frac{1}{N}\sum_{\widetilde{p},\widetilde{q}}\exp(i\frac{2\pi}{N}\sum_{r,\mu}\epsilon_{\mu\nu}d_\mu^{(p,q)}(r)\tilde{d}_\nu^{(\widetilde{p},\widetilde{q})}(\tilde{r}-\hat{\nu}))\tilde{Z}_V^{(\widetilde{p},\widetilde{q})}[\tilde{K},\tilde{d}]=$$

$$\frac{1}{N}\sum_{\tilde{p},\tilde{q}}\exp(i\frac{2\pi}{N}(p\widetilde{q}-q\widetilde{p}))\tilde{Z}_V^{(\widetilde{p},\widetilde{q})}[\tilde{K},\tilde{d}], \tag{25}$$

where

$$\tilde{Z}_V^{(\widetilde{p},\widetilde{q})}[\tilde{K},\tilde{d}]=\sum_{[\tilde{l}]}\exp(-\beta\tilde{H}^{(\widetilde{p},\widetilde{q})}[\tilde{l},\tilde{d}])$$

is the partition function of the model on the dual lattice.

Let us in detail consider the case $N=2$. Here Hamiltonian (8) coincides with Hamiltonian (4) of Ising model. In this case from (20) one gets

$$-\beta\tilde{H}_2^{(\widetilde{p},\widetilde{q})}[\tilde{l}]=\sum_{r,\nu}[g_\nu^{(0)}(\tilde{K})+g_\nu^{(1)}(\tilde{K})\cos\pi(\Delta_\nu^{\tilde{p}_\nu}\tilde{l}(r))].$$

In order to find coefficients $g_\mu^{(i)}(\tilde{K})$ we use the inverse Fourier transformation

$$\exp(\sum_k^M g_\mu^{(k)}(K)\cos^k\frac{2\pi}{N}\tilde{t}_\mu)=\frac{1}{N}\sum_{n=0}^{N-1}\exp(K_\nu\cos\frac{2\pi}{N}n-i\frac{2\pi}{N}n\tilde{t}_\nu). \tag{26}$$

Here $\mu\neq\nu$. Hence it is not hard to obtain for $N=2$:

$$e^{2g_\mu^{(0)}(\tilde{r})}=\frac{1}{2}\sinh 2K_\nu(r),\quad e^{-2g_\mu^{(1)}(\tilde{r})}=\tanh K_\nu(r)=e^{-2\tilde{K}_\mu(\tilde{r})},$$

where the last relation coinsides with (2). Using these relations, duality relation (25) one can represented in the form

$$\prod_{r,\mu}(\sinh 2K_\mu(r))^{-1/4}Z^{(p,q)}[K]=\frac{1}{2}\prod_{\tilde{r},\mu}(\sinh 2\tilde{K}_\mu(\tilde{r}))^{-1/4}\sum_{\tilde{p},\tilde{q}=0}^{1}e^{i\pi(p\tilde{q}-q\tilde{p})}\tilde{Z}^{(\tilde{p},\tilde{q})}[\tilde{K}], \tag{27}$$

where

$$\tilde{Z}^{(\widetilde{p},\widetilde{q})}[\tilde{K}]=\sum_{[l]}\exp\sum_{\tilde{r},\mu}(\tilde{K}_\mu(\tilde{r})\cos\pi(\Delta_\mu^{\tilde{p}_\mu}\tilde{l}(r))=\sum_{[\sigma]}\exp\sum_{\tilde{r},\mu}(\tilde{K}_\mu(\tilde{r})\tilde{\sigma}(\tilde{r})\nabla_\mu^{(\tilde{p}_\mu)}\tilde{\sigma}(\tilde{r})),$$

and $\tilde{\sigma}(\tilde{r})=\pm1$. It is easy to verify that (27) coinsides with duality relation (3), since the matrix

$$T^{p_x,p_y}_{\widetilde{p_x},\widetilde{p_y}}=T^{p,q}_{\widetilde{p},\widetilde{q}}=e^{i\pi(p\widetilde{q}-q\widetilde{p})}. \tag{28}$$

3 The $Z(N)$-Berezinsky-Villain model

Now let us consider the duality relation for the $Z(N)$-symmetric Berezinsky-Villain model. Partition function of this model one can write in the following form [6,7]:

$$Z_{BV}^{(p,q)}[K] = \sum_{[l]} e^{-\beta H_G^{(p,q)}[K,l]} = \sum_{[l]}\sum_{[k]}\prod_{r,\mu} \exp\left\{-\frac{1}{2}K_\mu(r)[\frac{2\pi}{N}\Delta_\mu l(r) - 2\pi k_\mu(r)]^2\right\}, \tag{29}$$

where

$$\sum_{[l]} = \prod_r(\sum_{l(r)=0}^{N-1}), \quad \sum_{[k]} = \prod_{r,\mu}(\sum_{k_\mu(r)=-\infty}^{\infty}).$$

Here $l(r) = 0, ..., N-1$ is on a site of the square lattice, index (p,q) defines the boundary conditions (10) ans the sum over k_μ guarantes the periodicity of the Hamiltonian relative to shifts $l \to l(r) + NL(r)$, where L is integer. By analogy with the partition function of the vector Potts model (29) one can rewrite in term of the basic magnetic dislocations $D^{(p,q)}$:

$$Z_{BV}^{(p,q)}[K,d] = \sum_{[l]}\sum_{[k]}\prod_{r,\mu} \exp\left\{-\frac{1}{2}K_\mu(r)[\frac{2\pi}{N}(\Delta_\mu l(r) + d_\mu^{(p,q)}(r)) - 2\pi k_\mu(r)]^2\right\}, \tag{30}$$

where $l(r)$ satisfies the periodical boundary conditions and the density $d_\mu(r)$ of the topological charge is determined by relations (12)-(14).

For derivation of the duality relation let us make the following transformations with (30):

$$Z_{BV}^{(p,q)}[K,d] = \sum_{[l]}\sum_{[k]}\prod_{r,\mu} \exp\left\{-\frac{1}{2}K_\mu(r)[\frac{2\pi}{N}(\Delta_\mu l(r) + d_\mu(r)) - 2\pi k_\mu(r)]^2\right\} = \tag{31}$$

$$(\prod_{r,\mu} N)^{\frac{1}{2}} \sum_{[s]}\sum_{[k]} \int D\theta \prod_{r,\mu} \exp\left\{-\frac{1}{2}K_\mu(r)[(\Delta_\mu\theta(r) + \frac{2\pi}{N}d_\mu(r)) - 2\pi k_\mu(r)]^2 + i\frac{N}{2}s(r)\theta(r)\right\} = \tag{32}$$

$$(\prod_{r,\mu} \frac{N}{2\pi K_\mu(r)})^{\frac{1}{2}} \sum_{[s]}\sum_{[t]} \int D\theta \prod_{r,\mu} \exp\left\{-\frac{1}{2K_\mu(r)}t_\mu^2(r) + it_\mu(r)(\Delta_\mu\theta(r) + \frac{2\pi}{N}d_\mu(r)) \right. \tag{33}$$

$$\left. + i\frac{N}{2}s(r)\theta(r)\right\} = (\prod_{r,\mu} \frac{N}{2\pi K_\mu(r)})^{\frac{1}{2}} \sum_{[s]}\sum_{[t]}\prod_{r,\mu} \exp\left\{-\frac{1}{2K_\mu(r)}t_\mu^2(r) + i\frac{2\pi}{N}t_\mu(r)d_\mu(r)\right\}$$

$$\prod_r \delta(\sum_\mu \Delta_\mu t_\mu(r-\widehat{\mu}) - Ns(r)), \tag{34}$$

where

$$\sum_{[t]} = \prod_{r,\mu}(\sum_{t_\mu(r)=-\infty}^{\infty}), \quad \int D\theta = \prod_r(\int_0^{2\pi} \frac{d\theta(r)}{2\pi}).$$

For derivation (32)-(34) we have used the summation formula

$$\frac{2\pi}{N}\sum_{l=0}^{N-1} \delta(\theta - \frac{2\pi}{N}l0) = \sum_{s=-\infty}^{\infty} e^{iNs\theta}, \quad 0 \le \theta \le 2\pi,$$

the identity

$$\sum_{k=-\infty}^{\infty} \exp[-\frac{1}{2}K(f-2\pi k)^2] = \frac{1}{\sqrt{2\pi K}} \sum_{t=-\infty}^{\infty} \exp(-\frac{1}{2K}t^2 + itf)$$

and the definition of δ-function

$$\int_0^{2\pi} \frac{d\theta(r)}{2\pi} e^{i\theta l} = \delta(l).$$

In order to take off the δ-function in (34) it is necessary to solve the equation

$$\sum_\mu \Delta_\mu t_\mu(r-\widehat{\mu}) = Ns(r). \tag{35}$$

Analysis of the solutions of this equation is similar to analysis of (21)-(24) and leads to the followig expressions for the gauge-nonequivalent solutions

$$t_\mu^{(\widetilde{p},\widetilde{q})}(r) = \epsilon_{\mu\nu}\Delta_\nu \tilde{l}(\tilde{r}-\widehat{\nu}) + \epsilon_{\mu\nu}\tilde{d}_\nu^{(\widetilde{p},\widetilde{q})}(\tilde{r}-\widehat{\nu}) - N\epsilon_{\mu\nu}\tilde{k}_\nu(\tilde{r}-\widehat{\nu}). \tag{36}$$

$$s(r) = \epsilon_{\mu\nu}\Delta_\mu \tilde{k}_\nu(\tilde{r}-\widehat{\nu}-\widehat{\mu}). \tag{37}$$

These solutions are the basic magnetic dislocations $\tilde{D}^{(\widetilde{p},\widetilde{q})}$ on the dual lattice in corresponding the gauge-nonequivalent classes $\tilde{\Omega}^{(\widetilde{p},\widetilde{q})}$ with the topological charge $Q_\mu = (\tilde{p}n, \tilde{q}m)$ $(\tilde{p},\tilde{q} = 0,1,...,N-1)$. Then, taking off δ-functions in (34), it is necessary to sum over all these solutions. In result we obtain the duality relation for the $Z(N)$-Berezinsky-Villain model

$$(\prod_{r,\mu} \frac{2\pi K_\mu(r)}{N})^{\frac{1}{4}} Z_{BV}^{(p,q)}[K,d] = \frac{1}{N}\sum_{\tilde{p},\tilde{q}} \exp(i\frac{2\pi}{N}(p\tilde{q}-q\tilde{p}))(\prod_{\widetilde{r},\mu} \frac{2\pi \widetilde{K}_\mu(\widetilde{r})}{N})^{\frac{1}{4}} \tilde{Z}_{BV}^{(\widetilde{p},\widetilde{q})}[\widetilde{K},\widetilde{d}], \tag{38}$$

where

$$K_\mu(r)\widetilde{K}_{-\nu}(\widetilde{r}) = (\frac{N}{2\pi})^2, \quad \mu \neq \nu.$$

As it is shown in [8] this model at $N=2$ corresponds to the Ising model, what is consistent with our result (38), which coincides with (3) in this case. V.S. thanks Dr. A. Morozov for the hospitality and the exellent conditions at ITEP, where this paper has been finished.

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